

Basic Numerical Procedures

Chapter 19

Approaches to Derivatives Valuation

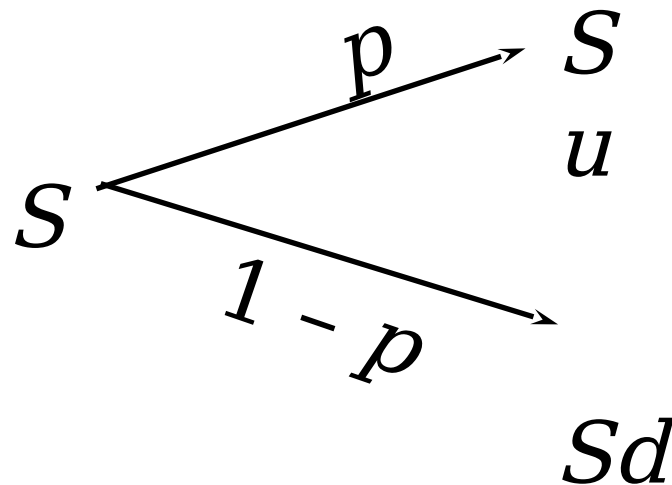
- Trees
- Monte Carlo simulation
- Finite difference methods

Binomial Trees

- Binomial trees are frequently used to approximate the movements in the price of a stock or other asset
- In each small interval of time the stock price is assumed to move up by a proportional amount u or to move down by a proportional amount d

Movements in Time Δt

(Figure 19.1, page 408)



Tree Parameters for asset paying a dividend yield of q

Parameters p , u , and d are chosen so that the tree gives correct values for the mean & variance of the stock price changes in a risk-neutral world

$$\text{Mean: } e^{(r-q)\Delta t} = pu + (1-p)d$$

$$\text{Variance: } \sigma^2 \Delta t = pu^2 + (1-p)d^2 - e^{2(r-q)\Delta t}$$

A further condition often imposed is $u = 1/d$

Tree Parameters for asset paying a dividend yield of q

(continued)

When Δt is small a solution to the equations is

$$u = e^{\sigma\sqrt{\Delta t}}$$

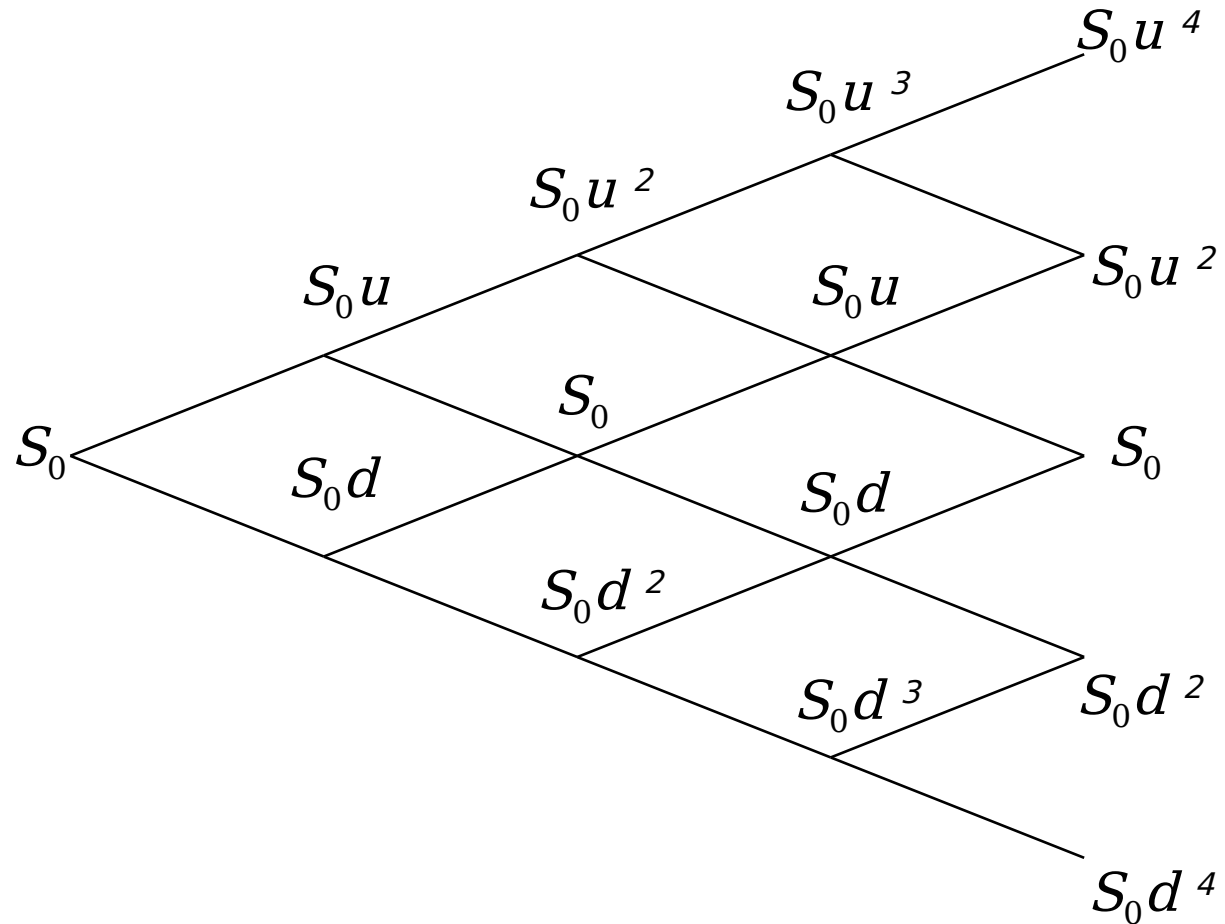
$$d = e^{-\sigma\sqrt{\Delta t}}$$

$$p = \frac{a - d}{u - d}$$

$$a = e^{(r - q)\Delta t}$$

The Complete Tree

(Figure 19.2, page 410)



Backwards Induction

- We know the value of the option at the final nodes
- We work back through the tree using risk-neutral valuation to calculate the value of the option at each node, testing for early exercise when appropriate

Example: Put Option

(Example 19.1, page 410)

$$S_0 = 50; \quad K = 50; \quad r = 10\%; \quad \sigma = 40\%;$$

$$T = 5 \text{ months} = 0.4167;$$

$$\Delta t = 1 \text{ month} = 0.0833$$

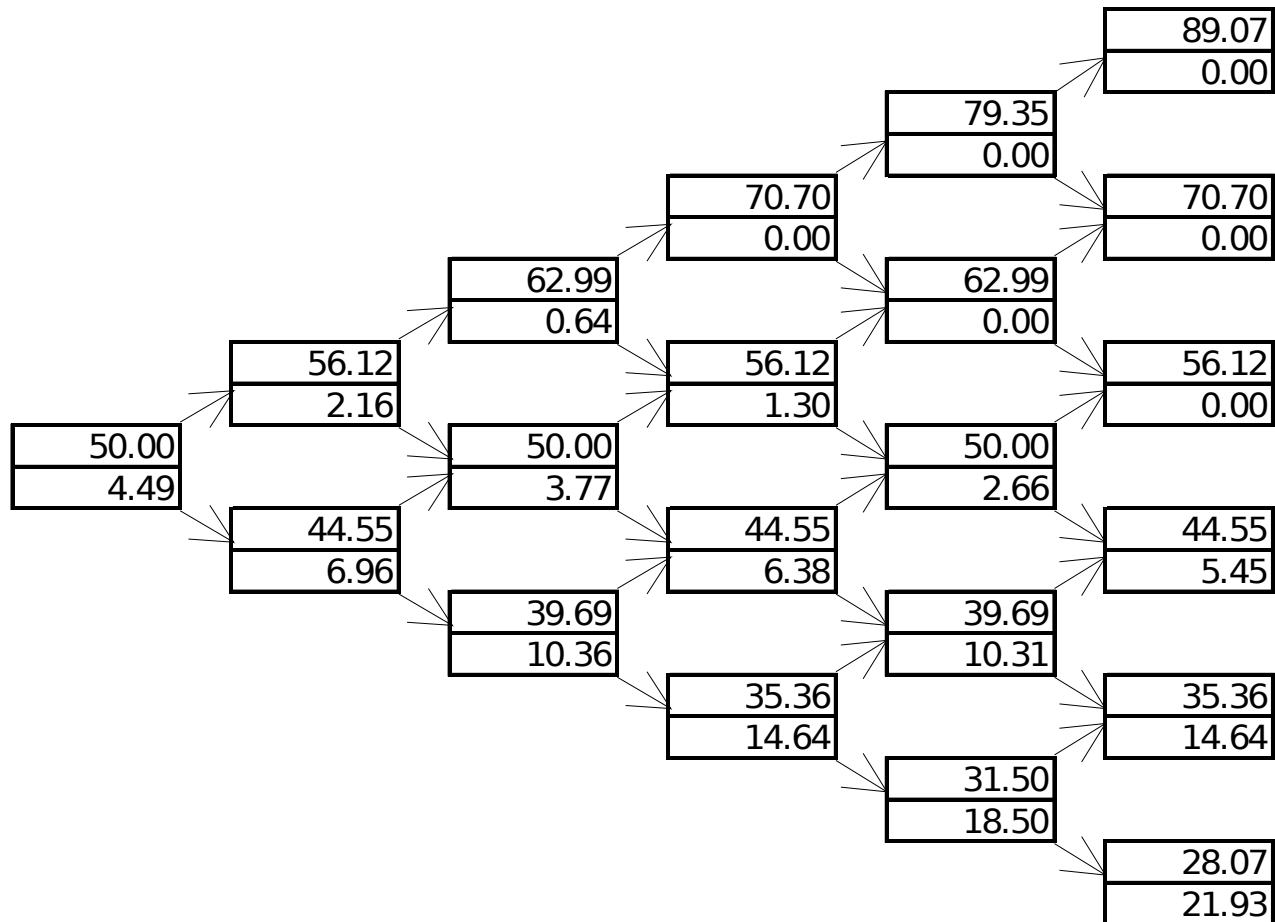
The parameters imply

$$u = 1.1224; \quad d = 0.8909;$$

$$a = 1.0084; \quad p = 0.5073$$

Example (continued)

Figure 19.3, page 411



Calculation of Delta

Delta is calculated from the nodes at time Δt

$$\text{Delta} = \frac{2.16 - 6.96}{56.12 - 44.55} = -0.41$$

Calculation of Gamma

Gamma is calculated from the nodes at time $2\Delta t$

$$\Delta_1 = \frac{0.64 - 3.77}{62.99 - 50} = -0.24; \quad \Delta_2 = \frac{3.77 - 10.36}{50 - 39.69} = -0.64$$

$$\text{Gamma} = \frac{\Delta_1 - \Delta_2}{11.65} = 0.03$$

Calculation of Theta

Theta is calculated from the central nodes at times 0 and $2\Delta t$

$$\text{Theta} = \frac{3.77 - 4.49}{0.1667} = -4.3 \text{ per year}$$

or -0.012 per calendar day

Calculation of Vega

- We can proceed as follows
- Construct a new tree with a volatility of 41% instead of 40%.
- Value of option is 4.62
- Vega is

$$4.62 - 4.49 = 0.13$$

per 1% change in volatility

Trees for Options on Indices, Currencies and Futures Contracts

As with Black-Scholes:

- For options on stock indices, q equals the dividend yield on the index
- For options on a foreign currency, q equals the foreign risk-free rate
- For options on futures contracts $q = r$

Binomial Tree for Stock Paying Known Dividends

- Procedure:
 - Draw the tree for the stock price less the present value of the dividends
 - Create a new tree by adding the present value of the dividends at each node
- This ensures that the tree recombines and makes assumptions similar to those when the Black-Scholes model is used

Extensions of Tree Approach

- Time dependent interest rates
- The control variate technique

Alternative Binomial Tree

(Section 19.4, page 422)

Instead of setting $u = 1/d$ we can set each of the 2 probabilities to 0.5 and

$$u = e^{(r - q - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}}$$

$$d = e^{(r - q - \sigma^2/2)\Delta t - \sigma\sqrt{\Delta t}}$$

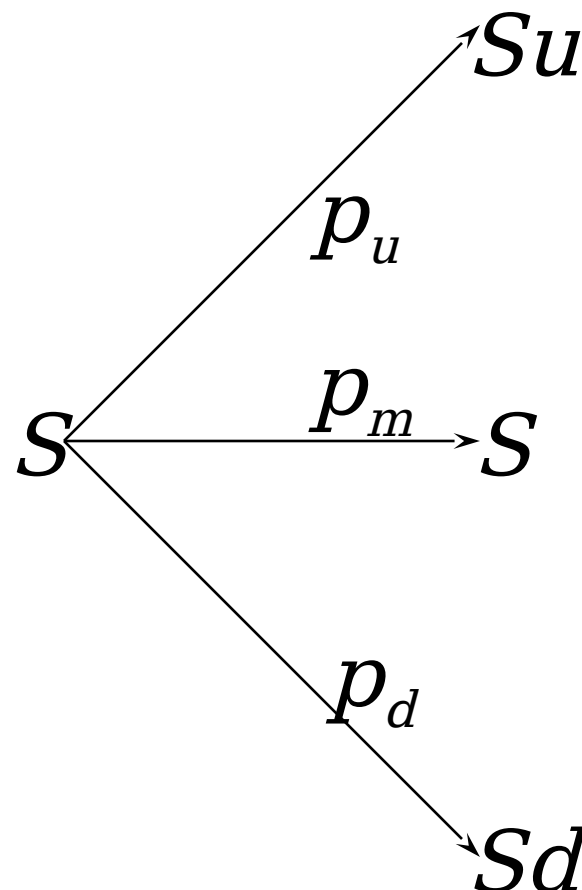
Trinomial Tree (Page 424)

$$u = e^{\sigma\sqrt{3\Delta t}} \quad d = 1/u$$

$$p_u = \sqrt{\frac{\Delta t}{12\sigma^2}} \left(r - \frac{\sigma^2}{2} \right) + \frac{1}{6}$$

$$p_m = \frac{2}{3}$$

$$p_d = -\sqrt{\frac{\Delta t}{12\sigma^2}} \left(r - \frac{\sigma^2}{2} \right) + \frac{1}{6}$$

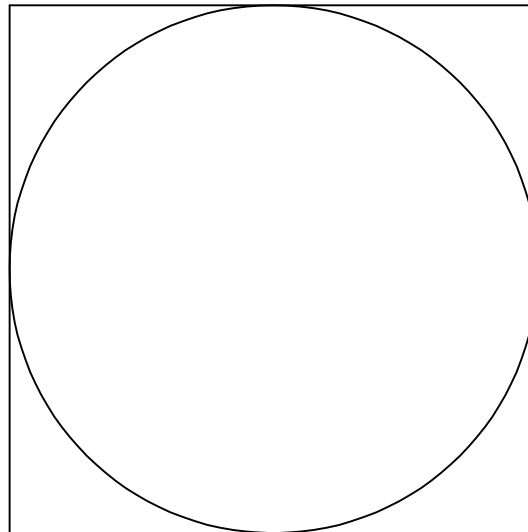


Time Dependent Parameters in a Binomial Tree (page 425)

- Making r or q a function of time does not affect the geometry of the tree. The probabilities on the tree become functions of time.
- We can make σ a function of time by making the lengths of the time steps inversely proportional to the variance rate.

Monte Carlo Simulation and π

- How could you calculate π by randomly sampling points in the square?



Monte Carlo Simulation and Options

When used to value European stock options, Monte Carlo simulation involves the following steps:

1. Simulate 1 path for the stock price in a risk neutral world
2. Calculate the payoff from the stock option
3. Repeat steps 1 and 2 many times to get many sample payoff
4. Calculate mean payoff
5. Discount mean payoff at risk free rate to get an estimate of the value of the option

Sampling Stock Price Movements

(Equations 19.13 and 19.14, page 427)

- In a risk neutral world the process for a stock price is

$$dS = \hat{\mu} S dt + \sigma S dz$$

- We can simulate a path by choosing time steps of length Δt and using the discrete version of this

$$\Delta S = \hat{\mu} S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

where ε is a random sample from $\phi(0,1)$

A More Accurate Approach

(Equation 19.15, page 428)

Use

$$d\ln S = (\hat{\mu} - \sigma^2/2) dt + \sigma dz$$

The discrete version of this is

$$\ln S(t + \Delta t) - \ln S(t) = (\hat{\mu} - \sigma^2/2) \Delta t + \sigma \varepsilon \sqrt{\Delta t}$$

or

$$S(t + \Delta t) = S(t) e^{(\hat{\mu} - \sigma^2/2) \Delta t + \sigma \varepsilon \sqrt{\Delta t}}$$

Extensions

When a derivative depends on several underlying variables we can simulate paths for each of them in a risk-neutral world to calculate the values for the derivative

Sampling from Normal Distribution (Page 414)

- One simple way to obtain a sample from $\phi(0,1)$ is to generate 12 random numbers between 0.0 & 1.0, take the sum, and subtract 6.0
- In Excel `=NORMSINV(RAND())` gives a random sample from $\phi(0,1)$

To Obtain 2 Correlated Normal Samples

- Obtain independent normal samples x_1 and x_2 and set

$$\begin{aligned}\varepsilon_1 &= x_1 \\ \varepsilon_2 &= \rho x_1 + x_2 \sqrt{1 - \rho^2}\end{aligned}$$

- Use a procedure known as Cholesky's decomposition when samples are required from more than two normal variables (see page 430)

Standard Errors in Monte Carlo Simulation

The standard error of the estimate of the option price is the standard deviation of the discounted payoffs given by the simulation trials divided by the square root of the number of observations.

Application of Monte Carlo Simulation

- Monte Carlo simulation can deal with path dependent options, options dependent on several underlying state variables, and options with complex payoffs
- It cannot easily deal with American-style options

Determining Greek Letters

For Δ :

1. Make a small change to asset price
2. Carry out the simulation again using the same random number streams
3. Estimate Δ as the change in the option price divided by the change in the asset price

Proceed in a similar manner for other Greek letters

Variance Reduction Techniques

- Antithetic variable technique
- Control variate technique
- Importance sampling
- Stratified sampling
- Moment matching
- Using quasi-random sequences

Sampling Through the Tree

Instead of sampling from the stochastic process we can sample paths randomly through a binomial or trinomial tree to value a derivative

Finite Difference Methods

- Finite difference methods aim to represent the differential equation in the form of a difference equation
- We form a grid by considering equally spaced time values and stock price values
- Define $f_{i,j}$ as the value of f at time $i\Delta t$ when the stock price is $j\Delta S$

Finite Difference Methods

(continued)

$$\text{In } \frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

$$\text{we set } \frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S}$$

$$\frac{\partial^2 f}{\partial S^2} = \left(\frac{f_{i,j+1} - f_{i,j}}{\Delta S} - \frac{f_{i,j} - f_{i,j-1}}{\Delta S} \right) / \Delta S \quad \text{or}$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta S^2}$$

Implicit Finite Difference Method

(Equation 19.25, page 437)

If we also set $\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t}$

we obtain the implicit finite difference method
This involves solving simultaneous equations
of the form:

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j}$$

Explicit Finite Difference Method

(page 439-444)

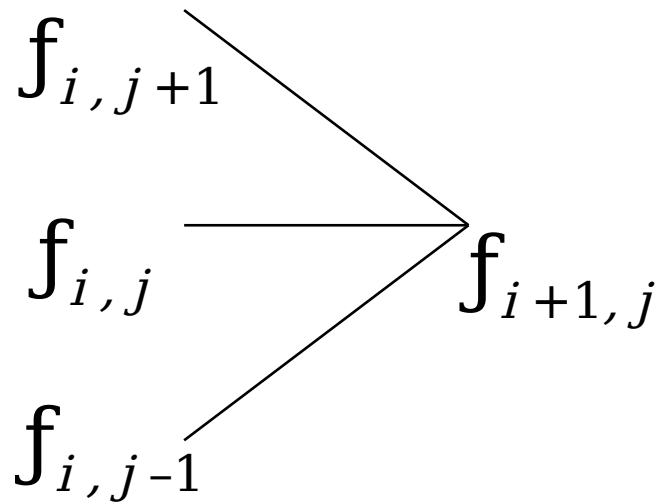
If $\partial f / \partial S$ and $\partial^2 f / \partial S^2$ are assumed to be the same at the $(i+1, j)$ point as they are at the (i, j) point we obtain the explicit finite difference method. This involves solving equations of the form:

$$f_{i,j} = a_j^* f_{i+1,j-1} + b_j^* f_{i+1,j} + c_j^* f_{i+1,j+1}$$

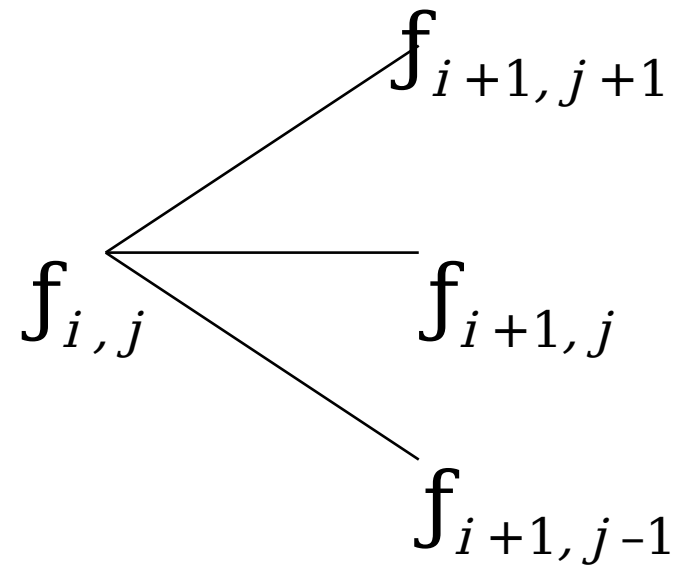
Implicit vs Explicit Finite Difference Method

- The explicit finite difference method is equivalent to the trinomial tree approach
- The implicit finite difference method is equivalent to a multinomial tree approach

Implicit vs Explicit Finite Difference Methods (Figure 19.16, page 441)



Implicit
Method



Explicit
Method

Other Points on Finite Difference Methods

- It is better to have $\ln S$ rather than S as the underlying variable
- Improvements over the basic implicit and explicit methods:
 - Hopscotch method
 - Crank-Nicolson method